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MULTIVARIATE CHARACTERISTICS OF RISK RUIN PROCESSES USING T-YEARS DEFERRED RUIN PROBABILITY

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ABSTRACT Frey and Schmidt(1996) obtained a recursive method of approximating finite time multivariate ruin probability based on a Mc-Laurin expansion for the classical case and exponentially tailed distributions of the claim size. In this work a generalization will be considered, first beyond the classical case and later, in the classical context, for any distribution of the claim size. It will be also proved that the recursive procedure can be simplified.

1. INTRODUCTION

Defining a classical risk process in continuous time $\{Z_t\}_{t\geq 0}$ with U_k claim sizes and premium c per time unit,

$$Z_t = u + ct - \sum_{k=1}^{N_t} U_k$$

where u are the initial reserves and N_t the total number of claims up to time t (with a c.d.f. of the waiting times between claims w(t)) where λ is the average number of claims in one year. Let B denote the distribution function of claim sizes U_k with mean $\mu^{-1}$and $c = \lambda\mu^{-1}(1+\theta)$, where θ is the premium loading factor.

Let us now define $\tau = \inf\{w > 0 : Z_w < 0\}$ as the ruin time and $Y = -Z_\tau$ as the deficit at ruin time or severity of ruin and $X = Z_{\tau^-}$ as the surplus just before the ruin.

We will now consider the concept of *t-years deferred ultimate ruin probability* with initial reserves u, severity of ruin less than y and surplus before ruin less than x ,

$$P\{\tau > t, X \leq x, Y \leq y\} =_{t|} \Psi_{u,x,y}$$

as the probability of the event consisting in that the stochastic process that models the reserves shall cross the ruin barrier, $Z_w < 0$ for the first time necessarily after t years.

The probability of ruin with time span t and initial reserves u, severity of ruin less than y and surplus before ruin less than x can be expressed,

$$P\{\tau < t, X \leq x, Y \leq y\} = \Psi_{t,u,x,y}$$

and the ultimate ruin probability,

$$P\{\tau < \infty, X \leq x, Y \leq y\} = \Psi_{u,x,y}$$

It is easy to prove that,

$$\Psi_{u,x,y} = \Psi_{t,u,x,y} +_{t|} \Psi_{u,x,y}$$

We will now prove that the t-years deferred ruin probability can be approximated recursively using total probability theorem and $\Psi_{z,x,y}$ for z$\in [0, u+ct]$.

Let us define the family of functions, t-years deferred ruin probability with exactly i claims in the interval (0,t]

$$\begin{aligned} {}_{t|}A^{i}_{u,x,y} &= P\{\tau > t, X \leq x, Y \leq y, N_t = i\} \\ i &= 0, 1, \ldots \end{aligned}$$

using total probability theorem over the number of claims N_t leads us to,

$$_{t|}\Psi_{u,x,y} = \sum_{i=0}^{\infty} {}_{t|}A^{i}_{u,x,y} \tag{1.1}$$

bearing in mind that the terms of the former series are probabilities, ${}_{t|}A^{i}_{u,x,y} \in [0,1]$, and

$$\lim_{i\to\infty} {}_{t|}A^{i}_{u,x,y} = 0$$

as it is obvious because the larger the number of claims in (0,t] the more likely the ruin to happen in that interval are suffient conditions of the convergence of the former series.

Let us see now how the members of the family of functions ${}_{t|}A^{i}_{u,x,y}$ (i=0,1,...) can be expressed recursively,

The first member of the family (no claims in (0,t], i=0) could be written as the probability of the joint event formed by two independent events:

– no claims in that interval :$(1 - W(t))$

– Ultimate ruin with initial reserves u+ct, starting at time point t,

$${}_{t|}A^{0}_{u,x,y} = \Psi_{u+ct,x,y}(1 - W(t))$$

The second member is obtained from the first one integrating over the time and the claim size,

$$
\begin{aligned}
{}_{t|}A^1_{u,x,y} &= \int_0^t \int_0^{u+cs} {}_{t-s|}A^0_{u+cs-z,x,y}\; b(z)\; w(s)\; dzds \\
&= \int_0^t \int_0^{u+cs} \Psi_{u+ct-z,x,y}\; b(z)\; w(s)\; (1-W(t-s))\; dzds
\end{aligned} \tag{1.2}
$$

if we define now the operator $\mathcal{I}({}_{t|}A^i_{u,x,y})$,

$$
\mathcal{I}({}_{t|}A^i_{u,x,y}) = \int_0^t \int_0^{u+cs} {}_{t-s|}A^i_{u+cs-z,x,y}\; b(z)\; w(s)\; dzds \tag{1.3}
$$

then

$$
{}_{t|}A^1_{u,x,y} = \mathcal{I}({}_{t|}A^0_{u,x,y})
$$

and subsequently,

$$
{}_{t|}A^i_{u,x,y} = \mathcal{I}({}_{t|}A^{i-1}_{u,x,y}) \qquad i = 1, 2, \ldots
$$

As a consequence, in order to approximate the t-deferred ruin probability with the series,

$$
{}_{t|}\Psi_{u,x,y} = \sum_{i=0}^{\infty} {}_{t|}A^i_{u,x,y}
$$

using the family of functions ${}_{t|}A^i_{u,x,y}$ (i=0,1,...) , we need to evaluate $\Psi_{z,x,y}$ for z$\in [0, u+ct]$ and proceed recursively using 1.3.

In the next sections we will study the Classical case of risk theory as a particular case within the framework stated before and compare with other results of actuarial literature related with approximating ${}_{t|}\Psi_{u,x,y}(\lambda)$ or $\Psi_{t,u,x,y}(\lambda)$.

2. Frey and Schmidt's approach

Frey and Schmidt(1996) obtained the following Taylor-Series expansion 2.2 for the probability of ruin with time span t, $\Psi_{t,u,x,y}$ when the distribution of the claim sizes has an exponential tail and the Classical Case of risk theory is considered, i.e. there are constants 0<c<∞ and $\gamma > 0$ such that $\lim_{x\to\infty} e^{\gamma x}(1 - B(x))$,and w(t)=$\lambda e^{-\lambda t}$, restricted to the case when the premium loading factor was defined,

$$
\theta = \frac{1-\lambda\mu^{-1}}{\lambda\mu^{-1}}; \qquad \theta > 0 \tag{2.1}
$$

$$
\Psi_{t,u,x,y}(\lambda) = \sum_{n=1}^{\infty} \frac{\Psi^{(n)}_{t,u,x,y}(0)}{n!}\lambda^n, \quad \lambda \geq 0 \tag{2.2}
$$

they proved that the function is analytic at $\lambda = 0$ and the Taylor-Series expansion 2.2 has an infinite radius of convergence. They also specified a recursive formula to obtain the n-th derivative at $\lambda = 0$(Theorem 2)

Theorem 1. *For each $n \geq 1$, $0 \leq u$, $t < \infty$, $0 < x$, $y \leq \infty$, it holds*

$$\frac{\Psi^{(n)}_{t,u,x,y}(0)}{n!} = \frac{\Psi^{(n)}_{u,x,y}(0)}{n!} - \sum_{k=1}^{n} q^{(n-k,k)}_{t,u,x,y} \tag{2.3}$$

where the quantities $q^{(n-k,k)}_{t,u,x,y}$ are given recursively by

$$q^{(n,k)}_{t,u,x,y} = \int_0^t \left(\int_0^{u+s} q^{(n-1,k)}_{t-s,u+s-z,x,y} dB(z) - q^{(n-1,k)}_{t-s,u+s,x,y} \right) ds \tag{2.4}$$

and

$$q^{(0,k)}_{t,u,x,y} = \frac{\Psi^{(k)}_{u+t,x,y}(0)}{k!}.$$

They also proved (in Theorem I) that

$$\frac{\Psi^{(k)}_{u+t,x,y}(0)}{k!} = F_{x,y} * G^{(k-1)}(u+t)$$

where $G(w) = \int_0^w (1 - B(z))dz$ and

$$F_{x,y}(w) = \int_w^{\max\{x,w\}} (B(z+y) - B(z))\, dz \tag{2.5}$$

then,

$$_{t|}\Psi_{u,x,y} = \sum_{n=1}^{\infty} \lambda^n \sum_{i=1}^{n} q^{(n-i,i)}_{t,u,x,y} \tag{2.6}$$

3. The Classical case

We will now consider the classical case of collective risk theory, exponential waiting time between claims, $\lambda e^{-\lambda t}$, in our approach,

$$\begin{aligned} {}_{t|}A^0_{u,x,y} &= \Psi_{u+t,x,y}(1 - W(t)) = e^{-\lambda t}\ \Psi_{u+t,x,y} \\ &= e^{-\lambda t}\ C^2_{t,u,x,y} \end{aligned}$$

$$
\begin{aligned}
{}_{t|}A^1_{u,x,y} &= \int_0^t \int_0^{u+cs} {}_{t-s|}A^0_{u+s-z,x,y}\ b(z)\ w(s)\ dzds \\
&= \lambda\ e^{-\lambda t} \int_0^t \int_0^{u+cs} \Psi_{u+t-z,x,y}\ b(z)\ dzds \\
&= \lambda\ e^{-\lambda t}\ C^3_{t,u,x,y}
\end{aligned}
$$

$$
\begin{aligned}
{}_{t|}A^2_{u,x,y} &= \int_0^t \int_0^{u+cs} {}_{t-s|}A^1_{u+s-z,x,y}\ b(z)\ w(s)\ dzds \\
&= \lambda^2\ e^{-\lambda t} \int_0^t \int_0^{u+cs} C^3_{t-s,u+t-z,x,y}(\lambda)\ b(z)\ dzds \\
&= \lambda^2\ e^{-\lambda t}\ C^4_{t,u,x,y}(\lambda)
\end{aligned}
$$

in general,

$$
\begin{aligned}
{}_{t|}A^k_{u,x,y} &= \int_0^t \int_0^{u+cs} {}_{t-s|}A^{k-1}_{u+s-z,x,y}\ b(z)\ w(s)\ dzds \\
&= \lambda^k\ e^{-\lambda t} \int_0^t \int_0^{u+cs} C^{k-1+2}_{t-s,u+t-z,x,y}\ b(z)\ dzds \\
&= \lambda^k\ e^{-\lambda t}\ C^{k+2}_{t,u,x,y}
\end{aligned}
$$

where

$$
\begin{aligned}
C^j_{t,u,x,y} &= \int_0^t \int_0^{u+cs} C^{j-1}_{t-s,u+s-z,x,y} b(z) dzds \\
j &= 3, 4, \ldots
\end{aligned}
$$

and

$$
C^2_{t,u,x,y} = \Psi_{u+ct,x,y}. \tag{3.1}
$$

It is clear then that the t-deferred ruin probability can be expressed,

$$
\begin{aligned}
{}_{t|}\Psi_{u,x,y} &= \sum_{k=0}^{\infty} {}_{t|}A^k_{u,x,y} \\
&= e^{-\lambda t} \sum_{k=0}^{\infty} \lambda^k\ C^{k+2}_{t,u,x,y}
\end{aligned} \tag{3.2}
$$

and after some easy arrangements in 3.2 we can obtain this alternative formula,

$$
\begin{aligned}
& {}_{t|}\Psi_{u,x,y} \\
&= e^{-\lambda t}\sum_{k=0}^{\infty}\lambda^k \, C^{k+2}_{t,u,x,y} = \sum_{k=0}^{\infty}\lambda^k\left(\sum_{j=0}^{\infty}(-1)^j \, \frac{(\lambda t)^j}{j!}\right) C^{k+2}_{t,u,x,y} \\
&= \sum_{k=0}^{\infty}\lambda^k\sum_{j=0}^{k}(-1)^j \, \left(\frac{t^j}{j!}\right) C^{k+2-j}_{t,u,x,y} \qquad (3.3)
\end{aligned}
$$

It is clear that for evaluating the family of functions $C^k_{t,u,x,y}$ $(k = 2, 3...)$ we need to obtain or approximate $\Psi_{z,x,y}$. Using a similar argument as in Gerber et al. (1987) we can use the following renewal equation for the ultimate ruin probability

$$\Psi_{z,x,y} = \frac{\lambda}{c}F_{x,y}(z) + \frac{\lambda}{c}\int_0^z \Psi_{z-\tau,x,y} \, dG(\tau)$$

where $G(w) = \int_0^w (1 - B(z))dz$ and

$$F_{x,y}(w) = \int_w^{\max\{x,w\}} (B(z+y) - B(z))\, dz \qquad (3.4)$$

and obtain the following power series expansion as Frey and Schmidt(1996) proved in Theorem 1,

$$\Psi_{z,x,y} = \sum_{n=1}^{\infty} \delta^n F_{x,y} * G^{*(n-1)}(z) \qquad (3.5)$$

where

$$\delta = \left(\frac{1}{(1+\theta)\mu^{-1}}\right)$$

Let us bear in mind that the ultimate ruin probability does not depend on the mean number of claims in the time unit λ, although Frey and Schmidt restricted their study to the case when the premium loading factor was defined,

$$\theta = \frac{1-\lambda\mu^{-1}}{\lambda\mu^{-1}}, \qquad \theta > 0$$

equation 3.5 holds for any value of this parameter.

If we now define recursively this family of functions,

$$p^{(2,k)}_{t,u,x,y} = F_{x,y} * G^{*(k-1)}(u+t)$$

$$\begin{aligned} p_{t,u,x,y}^{(i,k)} &= \int_0^t \int_0^{u+cs} p_{t-s,u+s-z,x,y}^{(i-1,k)} b(z)dz\, ds \\ i &= 3,4,... \end{aligned} \tag{3.6}$$

using the definition of $C_{t,u,x,y}^j$ and 3.5,

$$\begin{aligned} C_{t,u,x,y}^2 &= \Psi_{u+t,x,y} \\ &= \sum_{n=1}^{\infty} \delta^n F_{x,y} * G^{*(n-1)}(u+t) \\ &= \sum_{n=1}^{\infty} \delta^n p_{t,u,x,y}^{(2,n)} \end{aligned}$$

$$\begin{aligned} C_{t,u,x,y}^j &= \sum_{n=1}^{\infty} \delta^n p_{t,u,x,y}^{(j,n)} \\ j &= 3,4,... \end{aligned}$$

and substituting the former expressions in 3.2

$$\begin{aligned} {}_{t|}\Psi_{u,x,y} &= e^{-\lambda t} \sum_{k=0}^{\infty} \lambda^k\, C_{t,u,x,y}^{k+2} \\ &= e^{-\lambda t} \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \lambda^k\, \delta^n p_{t,u,x,y}^{(k+2,n)} \end{aligned} \tag{3.7}$$

if we now restrict ourselves to the case considered by Frey and Schmidt, $\lambda \equiv \delta$, in other words, $c = 1$,after some simple arrangements

$$\begin{aligned} {}_{t|}\Psi_{u,x,y} &= e^{-\lambda t} \sum_{k=1}^{\infty} \lambda^k \sum_{i=1}^{k} p_{t,u,x,y}^{(k+2-i,i)} \\ &= e^{-\lambda t} \sum_{k=1}^{\infty} \lambda^k \mathcal{F}_{t,u,x,y}^k \end{aligned} \tag{3.8}$$

where the functions $\mathcal{F}_{t,u,x,y}^k$, $k = 1,2,...$ can also be obtained recursively,

$$\begin{aligned} \mathcal{F}_{t,u,x,y}^k &= \sum_{i=1}^{k} p_{t,u,x,y}^{(k+2-i,i)} \\ \mathcal{F}_{t,u,x,y}^{k+1} &= \sum_{i=1}^{k+1} p_{t,u,x,y}^{((k+1)+2-i,i)} \\ &= \int_0^t \int_0^{u+cs} \mathcal{F}_{t,u,x,y}^k b(z)dz\, ds + p_{t,u,x,y}^{(2,k+1)} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\mathcal{F}^{1}_{t,u,x,y} &= p^{(2,i)}_{t,u,x,y} = F_{x,y} * G^{*(i-1)}(u+t)\\ i &= 1,2,\ldots\end{aligned}$$

which is a similar result as obtained by Schmidt and Frey, 2.6, but the recursive formulas are simpler (2.4 is more complicated than 3.9) and we do not need to restrict ourselves to exponentially tailed claims distributions.

4. Further proof of the equivalency of the formulas

We will prove now that the formulas obtained with our approach are equivalent to the formulas obtained by Frey and Schmidt(1996)

It is clear from 2.3 that the n-th term of the infinite sum 2.2 $\left(\frac{\Psi^{(n)}_{t,u,x,y}(0)}{n!}\lambda^n\right)$ is obtained from the n-th column of the following table, multiplying the first row by the sum of the rest of the members of that column.

λ	λ^2	λ^3	λ^4	$\ldots$
$\frac{\Psi^{(1)}_{u,x,y}(0)}{1!}$	$\frac{\Psi^{(2)}_{u,x,y}(0)}{2!}$	$\frac{\Psi^{(3)}_{u,x,y}(0)}{3!}$	$\frac{\Psi^{(4)}_{u,x,y}(0)}{4!}$	$\ldots$
$-q^{(0,1)}_{t,u,x,y}$	$-q^{(0,2)}_{t,u,x,y}$	$-q^{(0,3)}_{t,u,x,y}$	$-q^{(0,4)}_{t,u,x,y}$	$\ldots$
	$-q^{(1,1)}_{t,u,x,y}$	$-q^{(1,2)}_{t,u,x,y}$	$-q^{(1,3)}_{t,u,x,y}$	$\ldots$
		$-q^{(2,1)}_{t,u,x,y}$	$-q^{(2,2)}_{t,u,x,y}$	$\ldots$
			$-q^{(3,1)}_{t,u,x,y}$	$\ldots$
				$\ddots$

defining

$$column\,[j] = \lambda^j\left(\frac{\Psi^{(j)}_{u,x,y}(0)}{j!} - \sum_{k=1}^{j} q^{(j-k,k)}_{t,u,x,y}\right)$$

If we want to obtain the series 2.2 we wil need to use the whole information contained in the infinite number of columns, proceeding as stated above for each and every column and summing up the results,

$$\begin{aligned}\Psi_{t,u,x,y}(\lambda) &= \sum_{j=1}^{\infty} column\,[j]_{t,u,x,y}\,(\lambda)\\ &= \sum_{j=1}^{\infty}\lambda^j\left(\frac{\Psi^{(j)}_{u,x,y}(0)}{j!} - \sum_{k=1}^{j} q^{(j-k,k)}_{t,u,x,y}\right)\\ &= \sum_{n=1}^{\infty}\frac{\Psi^{(n)}_{t,u,x,y}(0)}{n!}\lambda^n\end{aligned}$$

Let us now sum along the different rows of table 1 instead of using the columns, we can define this family of functions

$$row\left[1\right]_{t,u,x,y}(\lambda)=\sum_{k=1}^{\infty}\frac{\Psi_{u,x,y}^{(n)}(0)}{k!}\lambda^{k}=\Psi_{u,x,y}(\lambda)$$

$$\begin{aligned}row\left[2\right]_{t,u,x,y}(\lambda) &= -\sum_{k=1}^{\infty}q_{t,u,x,y}^{(0,n)}\lambda^{k}\\ &= -\sum_{k=1}^{\infty}\frac{\Psi_{u+t,x,y}^{(n)}(0)}{k!}\lambda^{k}=-\Psi_{u+t,x,y}(\lambda)\end{aligned}$$

$$row\left[j\right]_{t,u,x,y}(\lambda)=-\lambda^{j-2}\sum_{k=1}^{\infty}q_{t,u,x,y}^{(j-2,k)}\lambda^{k}\quad j=3,\ldots \tag{4.1}$$

and as in the case of columns,

$$\begin{aligned}\Psi_{t,u,x,y}(\lambda) &= \sum_{k=1}^{\infty}\frac{\Psi_{t,u,x,y}^{(k)}(0)}{k!}\lambda^{k}\\ &= \sum_{k=1}^{\infty}row\left[k\right]_{t,u,x,y}(\lambda)\end{aligned}$$

or

$$\Psi_{t,u,x,y}(\lambda)=\Psi_{u,x,y}(\lambda)+\sum_{k=2}^{\infty}row\left[k\right]_{t,u,x,y}(\lambda)$$

finally

$$\begin{aligned}&\Psi_{u,x,y}(\lambda)-\Psi_{t,u,x,y}(\lambda)\\ &= {}_{t|}\Psi_{u,x,y}(\lambda)=-\sum_{k=2}^{\infty}row\left[k\right]_{t,u,x,y}(\lambda)\end{aligned} \tag{4.2}$$

Theorem 2. *Following the former approach let us prove this interesting recursive result,*

$$\begin{aligned}&row\left[j\right]_{t,u,x,y}(\lambda)\\ &= \lambda\mathcal{H}_1\left(row\left[j-1\right]_{t,u,x,y}(\lambda)\right)-\lambda\mathcal{H}_2\left(row\left[j-1\right]_{t,u,x,y}(\lambda)\right)\\ j &= 3,\ldots\end{aligned} \tag{4.3}$$

where the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined.

Lemma 3. **Theorem 4.**

$$\mathcal{H}_1\left(row\,[j-1]_{t,u,x,y}(\lambda)\right) = \int_0^t\int_0^{u+s} row\,[j-1]_{t-s,u+s-z,x,y}(\lambda)dB(z)ds$$
$$\mathcal{H}_2\left(row\,[j-1]_{t,u,x,y}(\lambda)\right) = \int_0^t row\,[j-1]_{t-s,u+s,x,y}(\lambda)ds$$

Proof.
using 2.4 and 4.1,

$$\begin{aligned}
row\,[j]_{t,u,x,y}(\lambda) &= -\lambda^{j-2}\sum_{k=1}^{\infty} q_{t,u,x,y}^{(j-2,k)}\lambda^k = -\lambda\left(\lambda^{((j-2)-1)}\sum_{k=1}^{\infty} q_{t,u,x,y}^{(j-2,k)}\lambda^k\right)\\
&= -\lambda\lambda^{((j-2)-1)}\\
&\quad \sum_{k=1}^{\infty}\left[\int_0^t\left(\int_0^{u+s} q_{t-s,u+s-z,x,y}^{((j-2)-1,k)}dB(z) - q_{t-s,u+s,x,y}^{((j-2)-1,k)}\right)ds\right]\lambda^k\\
&= \lambda\int_0^t\int_0^{u+s}\left[-\lambda^{((j-1)-2)}\sum_{k=1}^{\infty} q_{t-s,u+s-z,x,y}^{((j-1)-2,k)}\lambda^k\right]dB(z)ds\\
&\quad -\lambda\int_0^t\left[-\lambda^{((j-1)-2)}\sum_{k=1}^{\infty} q_{t-s,u+s,x,y}^{((j-1)-2,k)}\lambda^k\right]ds\\
&= \lambda\int_0^t\int_0^{u+s} row\,[j-1]_{t-s,u+s-z,x,y}(\lambda)dB(z)ds\\
&\quad -\lambda\int_0^t row\,[j-1]_{t-s,u+s,x,y}(\lambda)ds\\
j &= 3,\ldots
\end{aligned}$$

■

We can prove now that this result is just a particular case of the approach stated before when claims follow an exponential tailed distribution and θ, the premium loading factor, is defined in 2.1.

Lemma 5. *The following relation holds*

$$\begin{aligned}
&\Lambda_{t,u,x,y}^{i,j}(\lambda)\\
&= \left(\frac{1}{j!}\right)\sum_{l=0}^{j}\binom{j}{l}t^{j-l}\int_0^t(-s)^l\, C_{t-s,u+s,x,y}^{i-j}(\lambda)\,ds\\
&\quad -\frac{1}{(j+1)!}\sum_{l=0}^{j+1}\binom{j+1}{l}t^{(j+1)-l}
\end{aligned}$$

$$
\begin{aligned}
&\int_0^t (-s)^l \int_0^{u+s} C^{i-j-1}_{t-s,u+s-z,x,y}(\lambda) b(z) dz ds \\
= \quad & \frac{t^{j+1}}{(j+1)!} C^{i-j}_{t,u,x,y}(\lambda)
\end{aligned}
$$

where

$$
\begin{aligned}
C^j_{t,u,x,y}(\lambda) &= \int_0^t \int_0^{u+s} C^{j-1}_{t-s,u+s-z,x,y}(\lambda) b(z) dz ds \\
j &= 3, 4, \ldots
\end{aligned}
$$

and

$$
C^2_{t,u,x,y}(\lambda) = \Psi_{u+t,x,y}(\lambda)
$$

Proof.

$$
\begin{aligned}
& \left(\frac{1}{j!}\right) \sum_{l=0}^{j} \binom{j}{l} t^{j-l} \int_0^t (-s)^l \, C^{i-j}_{t-s,u+s,x,y}(\lambda) \, ds \\
& - \frac{1}{(j+1)!} \sum_{l=0}^{j+1} \binom{j+1}{l} t^{(j+1)-l} \\
& \int_0^t (-s)^l \int_0^{u+s} C^{i-j-1}_{t-s,u+s-z,x,y}(\lambda) b(z) dz ds \\
= \quad & \sum_{l=0}^{j} \left(\frac{1}{(j-l)!l!}\right) \, t^{j-l} \int_0^t (-s)^l \, C^{i-j}_{t-s,u+s,x,y}(\lambda) \, ds \\
& + \sum_{l=0}^{j+1} \left(\frac{1}{((j+1)-l)!l!}\right) t^{(j+1)-l} \int_0^t (-s)^l \int_0^{u+s} C^{i-j-1}_{t-s,u+s-z,x,y}(\lambda) b(z) dz ds \\
= \quad & \frac{t^{j+1}}{(j+1)!} C^{i-j}_{t,u,x,y}(\lambda) + \sum_{l=0}^{j} \left(\frac{1}{(j-l)!l!}\right) \, t^{j-l} \int_0^t (-s)^l \, C^{i-j}_{t-s,u+s,x,y}(\lambda) \, ds \\
& + \sum_{l=1}^{j+1} \left(\frac{1}{((j+1)-l)!l!}\right) t^{(j+1)-l} \int_0^t (-s)^l \int_0^{u+s} C^{i-j-1}_{t-s,u+s-z,x,y}(\lambda) b(z) dz ds \\
= \quad & \frac{t^{j+1}}{(j+1)!} C^{i-j}_{t,u,x,y}(\lambda) + \left(\frac{1}{(j-l)!l!}\right) \\
& \sum_{l=0}^{j} t^{j-l} \, (-1)^l \, [(l+1) \int_0^t s^l \, C^{i-j}_{t-s,u+s,x,y}(\lambda) \, ds \\
& - \int_0^t s^{(l+1)} \int_0^{u+s} C^{i-j-1}_{t-s,u+s-z,x,y}(\lambda) b(z) dz ds] \\
= \quad & \frac{t^{j+1}}{(j+1)!} C^{i-j}_{t,u,x,y}(\lambda)
\end{aligned}
$$

because,

$$
\begin{aligned}
& \int_0^t (l+1)\, s^l\, C^{i-j}_{t-s,u+s,x,y}(\lambda)\, ds \\
= & \int_{s=0}^t (l+1)\, s^l \int_{m=0}^{t-s} \int_{z=0}^{u+s+m} C^{i-j-1}_{t-s,u+s-z,x,y}(\lambda) b(z) dz dm\, ds \\
= & \int_{s=0}^t (l+1)\, s^l \int_{\varkappa=s}^{t} \int_{z=0}^{u+\varkappa} C^{i-j-1}_{t-s,u+s-z,x,y}(\lambda) b(z) dz d\varkappa\, ds \\
= & \int_{\varkappa=0}^t \int_{s=0}^{\varkappa} (l+1)\, s^l \int_{z=0}^{u+\varkappa} C^{i-j-1}_{t-s,u+s-z,x,y}(\lambda) b(z) dz ds\, d\varkappa \\
= & \int_{\varkappa=0}^t \varkappa^{l+1} \int_{z=0}^{u+\varkappa} C^{i-j-1}_{t-s,u+s-z,x,y}(\lambda) b(z) dz\, d\varkappa
\end{aligned}
$$

using the following change of variables,

$$
\begin{aligned}
m+s &= \varkappa \\
s &= s \\
z &= z
\end{aligned}
$$

and shifting the limits of the first two integrals.

■

Finally we will prove the following theorem

Theorem 6.

$$
\begin{aligned}
-row\,[i] &= \sum_{j=0}^{i-2} (-1)^j\, \lambda^{i-2-j}\, \frac{(\lambda t)^j}{j!} C^{i-j}_{t,u,x,y}(\lambda) \\
&= \lambda^{i-2} \sum_{j=0}^{i-2} (-1)^j\, \frac{t^j}{j!} C^{i-j}_{t,u,x,y}(\lambda) \qquad (4.4)
\end{aligned}
$$

where

$$
\begin{aligned}
C^j_{t,u,x,y}(\lambda) &= \int_0^t \int_0^{u+s} C^{j-1}_{t-s,u+s-z,x,y}(\lambda) b(z) dz ds \\
j &= 3, 4, \ldots
\end{aligned}
$$

and

$$
C^2_{t,u,x,y}(\lambda) = \Psi_{u+t,x,y}(\lambda)
$$

Proof. Let us proceed by complete induction,

$$
-row\,[2]_{t,u,x,y}(\lambda) = \Psi_{u+t,x,y}(\lambda) = C^2_{t,u,x,y}(\lambda)
$$

$$
\begin{aligned}
-row\,[3] &= \lambda\mathcal{H}_1\left(\Psi_{u+t,x,y}(\lambda)\right) \\
&\quad -\lambda\mathcal{H}_2\left(\Psi_{u+t,x,y}(\lambda)\right) \\
&= \lambda\int_0^t\int_0^{u+s}\Psi_{u+t-z,x,y}(\lambda)b(z)dzds \\
&\quad -\lambda\int_0^t\Psi_{u+t,x,y}(\lambda)ds \\
&= \lambda C^3_{t,u,x,y}(\lambda)-(\lambda t)\ C^2_{t,u,x,y}(\lambda)
\end{aligned}
$$

where

$$
C^3_{t,u,x,y}(\lambda)=\int_0^t\int_0^{u+s}\Psi_{u+t-z,x,y}(\lambda)b(z)dzds \tag{4.5}
$$

$$
\begin{aligned}
-row\,[4] &= \lambda\mathcal{H}_1\left(C^3_{t,u,x,y}(\lambda)\right)-\lambda\mathcal{H}_2\left(C^3_{t,u,x,y}(\lambda)\right) \\
&\quad -\lambda\mathcal{H}_1\left((\lambda t)\Psi_{u+t,x,y}(\lambda)\right)+\lambda\mathcal{H}_2\left((\lambda t)\Psi_{u+t,x,y}(\lambda)\right) \\
&= \lambda\int_0^t\int_0^{u+s}\lambda C^3_{t-s,u+s-z,x,y}(\lambda)b(z)dzds \\
&\quad -\lambda\int_0^t\lambda C^3_{t-s,u+s,x,y}(\lambda)ds \\
&\quad -\lambda\int_0^t(t-s)\int_0^{u+s}\lambda\Psi_{u+t-z,x,y}(\lambda)b(z)dzds \\
&\quad +\lambda\int_0^t\lambda(t-s)\Psi_{u+t,x,y}(\lambda)ds \\
&= \lambda^2\ C^4_{t,u,x,y}(\lambda)-\lambda\ (\lambda t)\ C^3_{t,u,x,y}(\lambda) \\
&\quad +\frac{(\lambda t)^2}{2!}C^2_{t,u,x,y}(\lambda)
\end{aligned}
$$

where

$$
C^4_{t,u,x,y}(\lambda)=\int_0^t\int_0^{u+s}C^3_{t-s,u+s-z,x,y}(\lambda)b(z)dzds
$$

Let us suppose now that 4.4 is true for $-row\,[i]_{t,u,x,y}$, then using 4.3,

$$
\begin{aligned}
&-row\,[i+1]_{t,u,x,y} \\
&= -\lambda\left(\mathcal{H}_1\left(row\,[i]_{t,u,x,y}(\lambda)\right)-\mathcal{H}_2\left(row\,[i]_{t,u,x,y}(\lambda)\right)\right) \\
&= \lambda\int_0^t\left(\sum_{j=0}^{i-2}(-1)^j\ \lambda^{i-2-j}\ \frac{(\lambda(t-s))^j}{j!}C^{i-j}_{t-s,u+s,x,y}(\lambda)\right)ds
\end{aligned}
$$

$$-\lambda \int_0^t \int_0^{u+s} \left(\sum_{j=0}^{i-2} (-1)^j \, \lambda^{i-2-j} \, \frac{(\lambda(t-s))^j}{j!} C^{i-j}_{t-s,u+s-z,x,y}(\lambda) \right) b(z)dz \, ds$$

$$= \sum_{j=0}^{i-2} (-1)^j \, \frac{\lambda^{(i+1)-2}}{j!} \int_0^t (t-s)^j \, C^{i-j}_{t-s,u+s,x,y}(\lambda)ds$$

$$-\sum_{j=0}^{i-2} (-1)^j \, \frac{\lambda^{(i+1)-2}}{j!} \int_0^t \int_0^{u+s} (t-s)^j \, C^{i-j}_{t-s,u+s-z,x,y}(\lambda)b(z)dz \, ds$$

$$= \lambda^{(i+1)-2} \, (-1)^{i-2} \int_0^t \frac{(t-s)^{i-2}}{(i-2)!} \, C^2_{t-s,u+s,x,y}(\lambda)ds$$

$$-\lambda^{(i+1)-2} \int_0^t \int_0^{u+s} C^i_{t-s,u+s-z,x,y}(\lambda)b(z)dz \, ds$$

$$+\lambda^{(i+1)-2} \sum_{j=0}^{i-3} \frac{(-1)^j}{j!} \int_0^t (t-s)^j \, C^{i-j}_{t-s,u+s,x,y}(\lambda)d$$

$$-\lambda^{(i+1)-2} \sum_{j=0}^{i-3} \frac{(-1)^{j+1}}{(j+1)!} \int_0^t \int_0^{u+s} (t-s)^{j+1} \, C^{i-(j+1)}_{t-s,u+s-z,x,y}(\lambda)b(z)dz \, ds$$

$$= \lambda^{(i+1)-2} \frac{(-1)^{i-2} t^{(i+1)-2}}{((i+1)-2)!} \Psi_{u+t,x,y}(\lambda) - \lambda^{(i+1)-2} C^{i+1}_{t,u,x,y}(\lambda)$$

$$+\lambda^{(i+1)-2} \sum_{j=0}^{i-3} (-1)^j \Lambda^{i,j}_{t,u,x,y}(\lambda)$$

$$= \lambda^{(i+1)-2} \Big(\frac{(-1)^{i-2} t^{(i+1)-2}}{((i+1)-2)!} \Psi_{u+t,x,y}(\lambda) + \sum_{j=0}^{i-3} (-1)^j \frac{t^{j+1}}{(j+1)!} C^{i-j}_{t,u,x,y}(\lambda)$$

$$-C^{i+1}_{t,u,x,y}(\lambda) \Big)$$

$$= \lambda^{(i+1)-2} \sum_{j=0}^{(i+1)-2} \frac{(-1)^j t^j}{j!} C^{(i+1)-j}_{t,u,x,y}(\lambda)$$

$$= \sum_{j=0}^{(i+1)-2} (-1)^j \, \lambda^{(i+1)-2-j} \, \frac{(\lambda t)^j}{j!} C^{(i+1)-j}_{t,u,x,y}(\lambda)$$

■

As a corolary it is easy to prove that,

$$
\begin{aligned}
{}_{t|}\Psi_{u,x,y}(\lambda) &= -\sum_{k=2}^{\infty} row\,[k]_{t,u,x,y}\,(\lambda) \\
&= \sum_{k=2}^{\infty} \lambda^{k-2} \sum_{j=0}^{k-2} (-1)^j \, \frac{t^j}{j!} C^{k-j}_{t,u,x,y}(\lambda)
\end{aligned}
$$

$$= \sum_{k=2}^{\infty} \lambda^{k-2} \left(\sum_{j=0}^{\infty} (-1)^j \frac{(\lambda t)^j}{j!} \right) C^k_{t,u,x,y}(\lambda)$$

$$= \sum_{k=2}^{\infty} \lambda^{k-2} e^{-\lambda t} C^k_{t,u,x,y}(\lambda)$$

5. Concluding Comments

The approach stated in this paper can be used with expression 1.1 for any distribution of the waiting times between claims with the only restriction of stationarity.

For the classical case, with formula 3.7 we can approximate the ruin probability in the multivariate case for any value of θ and μ^{-1}.

Even in the restricted case considered by Frey and Schmidt(1996): c=1 and claim sizes distributions have an exponential tail, our approach leads to simpler recursive formulas, **2.4 is more complicated than 3.9.**

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